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Vacuum birefringence in a rotating magnetic field

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Abstract

We calculate the vacuum polarization-induced ellipticity acquired by a linearly polarized laser beam of angular frequency $\bar{\omega}$ on traversing a region containing a transverse magnetic field rotating with a small angular velocity Ω around the beam axis. The transmitted beam contains the fundamental frequency $\bar{\omega}$ and weak sidebands of frequency $\bar{\omega} \pm 2\Omega$, but no other sidebands. To first order in small quantities, the ellipticity acquired by the transmitted beam is independent of Ω, and is the same as would be calculated in the approximation of regarding the magnetic field as fixed at its instantaneous angular orientation, using the standard vacuum birefringence formulae for a static magnetic field. Also to first order, there is no rotation of the polarization plane of the transmitted beam. Analogous statements hold when the magnetic field strength is slowly varying in time.

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1. Introduction

There has recently been renewed interest in trying to measure the birefringence of the vacuum in a strong magnetic field predicted by the Heisenberg–Euler effective Lagrangian [1–4] for quantum electrodynamics (QED). Such a measurement was a primary motivation for the PVLAS experiment [5], which however has reported a signal about 10^4 times larger than expected from QED. In a recent article [6], a claim was made that just the kinematics of rotation of the magnetic field used in the PVLAS experiment, through the generation of multiple sidebands to the transmitted laser beam, could explain the PVLAS result. However, examination of [6] and a related paper [7] suggests several problems[1]. First, the authors have overestimated the strength of the vacuum birefringence coefficient by a factor of over 1000. Second, since on bases rotating with the magnetic field, the vacuum as modified by the magnetic field still acts on an incident electromagnetic wave as a time-independent

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[1] Problems with the calculation of [6] were first pointed out by Holger Gies (unpublished memorandum, July 2006), and problems with [6, 7] were noted independently by the author (unpublished memorandum, October 2006).

linear medium, the presence of an infinite sequence of sidebands (as would be obtained from a nonlinear medium) is not expected; the only sidebands should be those associated with transformations between fixed laboratory bases and the rotating bases.

Nonetheless, the authors of [6] have called attention to an interesting, and as it turns out quite nontrivial, problem in wave propagation. This paper is devoted to formulating and solving this problem to leading order, using standard scattering theory and perturbation theory methods. Our conclusion is that to leading order in small quantities, the method used by the PVLAS group to calculate the birefringence-induced ellipticity yields the correct answer, which is independent of the rotation angular velocity Ω.

To set the stage, we start from the lowest order formulae describing electromagnetic wave propagation in an external magnetic field, as deduced from the Heisenberg–Euler effective Lagrangian. With repeated indices summed, the $\vec{d}$ and $\vec{h}$ fields are obtained from the $\vec{e}$ and $\vec{b}$ fields by

$$d_s = \epsilon_{st} e_t, \qquad h_s = \mu_{st}^{-1} b_t. \tag{1a}$$

The polarization tensors ϵ_{st} and μ_{st}^{-1} arising from the fourth-order box diagram are given by

$$\begin{aligned} \epsilon_{st} &= \delta_{st}(1 - 2\xi) + 7\xi \hat{B}_s(t)\hat{B}_t(t), \\ \mu_{st}^{-1} &= \delta_{st}(1 - 2\xi) - 4\xi \hat{B}_s(t)\hat{B}_t(t), \end{aligned} \tag{1b}$$

with $\hat{B}_s(t)$ being a unit vector along the magnetic field, which we take to have fixed magnitude B. With B measured in unrationalized units, and setting $\hbar = c = 1$, the parameter ξ governing the strength of the QED vacuum polarization effect is given by

$$\xi = \frac{\alpha^2 B^2}{45\pi m^4}, \tag{1c}$$

where $\alpha \simeq 1/137.04$ is the fine structure constant and m is the electron mass. Finally, in terms of $\vec{d}, \vec{h}, \vec{e}$ and $\vec{b}$ the Maxwell equation take the form

$$\begin{aligned} \vec{\nabla} \cdot \vec{d} &= \vec{\nabla} \cdot \vec{b} = 0, \\ \vec{\nabla} \times \vec{e} &= -\frac{\partial \vec{b}}{\partial t}, \qquad \vec{\nabla} \times \vec{h} = \frac{\partial \vec{d}}{\partial t}. \end{aligned} \tag{1d}$$

Equations (1a)–(1d) are the fundamental physical equations governing the wave propagation problem under study. When the magnetic field orientation $\hat{B}(t)$ is time independent, analysis of this problem shows that the vacuum becomes birefringent. Specializing to the case of a wave vector $\vec{k}$ perpendicular to $\hat{B}$, the polarization state with $\vec{e}$ perpendicular to $\hat{B}$ propagates with the index of refraction $k/\omega = 1 + 2\xi$, while the polarization state with $\vec{e}$ parallel to $\hat{B}$ propagates with the index of refraction $k/\omega = 1 + \frac{7}{2}\xi$ (with $k = |\vec{k}|$). We will denote the difference between the larger and smaller refractive indices by $\Delta n = \frac{3}{2}\xi$.

To finish specifying the problem, we shall assume that the linearly polarized laser beam is propagating upwards along the z axis, with field strengths (expressed in terms of (x, y, z) components) given by

$$\begin{aligned} \vec{e} = \vec{d} &= e^{i\bar{\omega}(z-t)}(\cos\theta, \sin\theta, 0), \\ \vec{b} = \vec{h} &= e^{i\bar{\omega}(z-t)}(-\sin\theta, \cos\theta, 0). \end{aligned} \tag{2a}$$

We assume that uniform magnetic field is confined to the region $0 \leqslant z \leqslant L$, and is oriented perpendicular to the z axis, and rotates uniformly with the angular velocity Ω, according to

$$\hat{B}(t) = (\cos\Omega t, \sin\Omega t, 0), \tag{2b}$$

from which we learn that (with the prime denoting time differentiation)

$$\hat{B}'(t) = \Omega(-\sin\Omega t, \cos\Omega t, 0) = \Omega\hat{z} \times \hat{B}. \tag{2c}$$

The problem is then to calculate the upward moving transmitted wave emerging at $z = L$.

Before proceeding with details, we outline our basic strategy. The first observation is that since equation ($2c$) relates the time derivative of $\hat{B}$ to $\hat{z} \times \hat{B}$ with a time-independent coefficient, the wave propagation problem in the region $0 \leqslant z \leqslant L$ will involve differential equations with time-independent coefficients when referred to the orthonormal rotating bases $\hat{B}$ and $\hat{z} \times \hat{B}$. These differential equations can then be solved by a standard travelling wave ansatz; this calculation of the propagation modes in the rotating magnetic field 'medium' is carried out in section 2. One then has to transform the incident wave to the rotating bases, and do a matching of incident, transmitted, and reflected waves at the medium boundaries $z = 0$ and $z = L$. This calculation is carried out in section 3. The calculation is greatly facilitated by working to leading order in the small quantities ξ, $\Omega/\bar{\omega}$ and ΩL, that is, we regard both ξ and Ω as small. Since each reflected wave is reduced in strength by $O(\xi)$, this allows us to neglect multiple reflections, such as a wave reflected back from $z = L$, and then reflected forward from $z = 0$, and finally emerging at $z = L$, which makes a contribution of order ξ^2 to the transmitted wave. When multiple reflections are neglected, one can solve the wave propagation problem (to leading order) by considering independent matching problems at $z = 0$ and $z = L$. Moreover, it then suffices to compute the phases of the various waves to first order in small quantities, but the corresponding transmitted wave amplitudes only to zeroth order. The reason this approximation is adequate is that the transmission coefficients at the two boundaries are independent of L when multiple reflections are neglected, and so first-order terms in the transmission coefficients can contribute only an L-independent term to the polarization parameters of the emerging wave at $z = L$. However, when $L = 0$ no birefringent medium is traversed, and so the ellipticity of the emerging wave is zero; hence the first-order corrections to the wave coefficients must make a vanishing contribution to the ellipticity, and similarly, to the rotation of the polarization axis. The calculation of the ellipticity and polarization axis, from our formulae for the emerging waves, is given in section 4. An alternative calculation method, based on a perturbative expansion in ξ, is given in section 5, which gives results identical to the wave-matching calculation of section 3 for the rotating field case, and also applies to the case when the magnetic field strength is time dependent. A brief discussion of our results and their experimental implications is given in section 6.

2. Electromagnetic wave eigenmodes in a rotating magnetic field

We begin by calculating the electromagnetic wave eigenmodes in the region containing the rotating magnetic field. We write the $\vec{e}$ and $\vec{b}$ fields in terms of components along the rotating bases $\hat{B}$ and $\hat{z} \times \hat{B}$,

$$\vec{e} = [E_1(t)\hat{B} + E_2(t)\hat{z} \times \hat{B}]\,\mathrm{e}^{\mathrm{i}z}, \qquad \vec{b} = [B_1(t)\hat{B} + B_2(t)\hat{z} \times \hat{B}]\,\mathrm{e}^{\mathrm{i}z}. \tag{3a}$$

Substituting these into equation ($1a$), and using equation ($1b$), we get the corresponding expressions for the $\vec{d}$ and $\vec{h}$ fields,

$$\vec{d} = [(1 + 5\xi)E_1(t)\hat{B} + (1 - 2\xi)E_2(t)\hat{z} \times \hat{B}]\,\mathrm{e}^{\mathrm{i}z},$$
$$\vec{h} = [(1 - 6\xi)B_1(t)\hat{B} + (1 - 2\xi)B_2(t)\hat{z} \times \hat{B}]\,\mathrm{e}^{\mathrm{i}z}. \tag{3b}$$

Substituting equations ($3a$), ($3b$) into the Maxwell equations of equation ($1d$), and using equation ($2c$), we get a set of coupled equations for the coefficient functions $E_{1,2}(t)$ and

$B_{1,2}(t)$,

$$ik E_1(t) = -\Omega B_1(t) - B_2'(t),$$
$$-ik E_2(t) = -B_1'(t) + \Omega B_2(t).$$
$$ik(1 - 6\xi)B_1(t) = (1 + 5\xi)\Omega E_1(t) + (1 - 2\xi)E_2'(t),$$
$$-ik(1 - 2\xi)B_2(t) = (1 + 5\xi)E_1'(t) - (1 - 2\xi)\Omega E_2(t).$$
$$(4a)$$

Since this set of equations has time-independent coefficients, it can be solved by making an exponential ansatz,

$$E_{1,2}(t) = E_{1,2}\,\mathrm{e}^{-\mathrm{i}\omega t}, \qquad B_{1,2}(t) = B_{1,2}\,\mathrm{e}^{-\mathrm{i}\omega t}. \qquad (4b)$$

Substituting equation (4b) into equation (4a), and simplifying by neglecting second-order small terms proportional to ξ^2 and $\xi\Omega$, we get the coupled equations for the constant amplitudes $E_{1,2}$ and $B_{1,2}$,

$$kE_1 = \omega B_2 + \Omega\,\mathrm{i}B_1, \qquad\qquad k\,\mathrm{i}E_2 = -\omega\,\mathrm{i}B_1 - \Omega B_2,$$
$$k\,\mathrm{i}B_1 = -\omega(1 + 4\xi)\,\mathrm{i}E_2 + \Omega E_1, \qquad kB_2 = \omega(1 + 7\xi)E_1 - \Omega\,\mathrm{i}E_2.$$
$$(4c)$$

We note at this point that we have ignored effects arising from the electric field induced by the rotating magnetic field; these are of order $\xi\Omega$ at least, and so are second order in the small parameters of the problem. Similarly, other relativistic effects associated with electrodynamics in rotating frames, on which there is a substantial literature, are not relevant for our analysis.

Since equation (4c) is a homogeneous set of linear equations, a solution is possible only when the determinant vanishes, which yields a quartic equation giving the dispersion relation of the wave

$$[k^2 - \omega^2(1 + 7\xi)][k^2 - \omega^2(1 + 4\xi)] - 2\Omega^2(\omega^2 + k^2) + \Omega^4 = 0. \qquad (5a)$$

This can be solved to give two solutions $k_\pm$ which describe upward propagating waves, and two solutions $-k_\pm$ which describe downward propagating waves, with $k_\pm$ given by

$$k_\pm = \omega \pm \Omega + \tfrac{1}{2}\omega\sigma_\pm,$$
$$\sigma_\pm \equiv \sigma_\pm(\omega) = \tfrac{11}{2}\xi \pm \left[((2\Omega/\omega)^2 + (3\xi/2)^2)^{\frac{1}{2}} - 2\Omega/\omega\right],$$
$$(5b)$$

so that the corresponding refractive indices are given by

$$k_\pm/\omega = n_\pm = 1 \pm \frac{\Omega}{\omega} + \frac{1}{2}\sigma_\pm. \qquad (5c)$$

Note that $\sigma_\pm$ is first order in small quantities, and is uniformly of order ξ or smaller irrespective of the relative sizes of ξ and Ω/ω. Also, we note that the terms of order $\xi\Omega$ that we have dropped make a contribution inside the square root of order $\xi(\Omega/\omega)^2$ (because the dispersion relation must be an even function of Ω, as we have verified by explicit computation), and so is a higher order effect relative to the quadratic terms that we retained inside the square root.

Solving for the ratios of the field coefficients to E_1, we get for the case of upward propagating waves with oscillatory behaviour $\mathrm{e}^{\mathrm{i}(n_\pm z - t)\omega}$

$$\frac{B_2}{E_1} = 1 + \beta_\pm, \qquad \frac{\mathrm{i}E_2}{E_1} = \mp 1 + \alpha_\pm, \qquad \frac{\mathrm{i}B_1}{E_1} = \pm 1 + \gamma_\pm, \qquad (6a)$$

with the quantities $\alpha_\pm$, $\beta_\pm$, $\gamma_\pm$, which all vanish when ξ vanishes, given by

$$\beta_\pm = \frac{7}{2}\xi,$$

$$\alpha_\pm = \frac{\omega}{2\Omega}(1 - 2\xi)(7\xi - \sigma_\pm) \pm 2\xi$$

$$\simeq \frac{\omega}{2\Omega}(7\xi - \sigma_\pm), \qquad (6b)$$

$$\gamma_\pm = -\frac{\omega}{2\Omega}(2 + 2\xi - n_\pm)(7\xi - \sigma_\pm) \pm \left(2\xi - \frac{1}{2}\sigma_\pm\right)$$

$$\simeq -\alpha_\pm,$$

where the first lines of the expressions for $\alpha_\pm$ and $\gamma_\pm$ are accurate to first order in small quantities, and the second lines are accurate to leading zeroth order. For the case of downward propagating waves with oscillatory behaviour $e^{i(-n_\pm z - t)\omega}$, one simply reverses the sign of B_2/E_1 and iB_1/E_1, while keeping iE_2/E_1 the same as in equations (6a), (6b). This completes the analysis of the electromagnetic wave eigenmodes in the region $0 \leqslant z \leqslant L$ containing the rotating magnetic field B.

3. Wave matching to get the transmission coefficient

Now that we know the form of propagating waves in the magnetic field region, we turn to the problem of determining the emerging waves at $z \geqslant L$ produced by the laser beam incident from below $z = 0$. As noted in section 1, when doubly reflected waves are ignored, we can solve this problem by treating sequentially the wave matching at $z = 0$, followed by the wave matching at $z = L$. The first step is to rewrite the incident wave at $z = 0$ on the rotating basis, giving

$$\vec{e}\,|_{z=0} = e^{-i\bar{\omega}t}[\cos(\theta - \Omega t)\hat{B} + \sin(\theta - \Omega t)\hat{z} \times \hat{B}],$$
$$\vec{b}\,|_{z=0} = e^{-i\bar{\omega}t}[-\sin(\theta - \Omega t)\hat{B} + \cos(\theta - \Omega t)\hat{z} \times \hat{B}], \qquad (7)$$

which shows that all the components of the incident wave on the rotating bases have the time dependence $e^{-i(\bar{\omega}+\Omega)t}$ or $e^{-i(\bar{\omega}-\Omega)t}$. Thus to do the match, we need a reflected wave at $z \leqslant 0$ with four coefficients (two for the reflected wave with frequency $\bar{\omega} + \Omega$, and two for the reflected wave with frequency $\bar{\omega} - \Omega$), and a transmitted wave with four coefficients (one each for the $\pm$ eigenmodes with frequency $\omega = \bar{\omega} + \Omega$, and one each for the $\pm$ eigenmodes with frequency $\omega = \bar{\omega} - \Omega$). Continuity of the $\vec{e}$ and $\vec{h}$ fields across the junction at $z = 0$ gives eight matching conditions (there are two field components parallel to the boundary for each of the two fields to be matched, and two frequency components for each). Solving the wave matching equations shows, as expected, that the reflected wave amplitudes are of order ξ, so to get the transmitted wave amplitudes to zeroth order it suffices to ignore the reflected waves. Similarly, at $z = L$ there are also eight unknown coefficients, four for the transmitted wave (two for frequency $\bar{\omega} + \Omega$ and two for frequency $\bar{\omega} - \Omega$), and four for the waves of the two frequencies reflected back into the magnetic field region $z \leqslant L$. Again there are eight matching conditions, and solving for the reflected waves shows that they are again of order ξ in magnitude, and so again the leading-order coefficients of the transmitted wave can be obtained by ignoring the reflected waves.

Thus, the transmitted wave on the rotating basis, to leading order in the wave amplitudes, is simply the result of compounding transmission at $z = 0$, followed by propagation from $z = 0$ to $z = L$, and then followed by transmission at $z = L$. The final step is to convert back from the rotating bases to fixed laboratory bases. This yields three distinct emerging wave

components, with frequencies $\bar{\omega}$, $\bar{\omega}+2\Omega$ and $\bar{\omega}-2\Omega$. We dispense with all of the intermediate algebra, and simply present the final result for the transmitted wave at $z = L$, which takes the form

$$\vec{e}|_{z=L} = e^{i\bar{\omega}(L-t)}(X, Y, 0), \qquad \vec{b}|_{z=L} = e^{i\bar{\omega}(L-t)}(-Y, X, 0), \tag{8a}$$

with the components X, Y given in terms of quantities A, B, C, D by

$$X = \frac{1}{2}(C + D + A\,e^{2i\Omega t} + B\,e^{-2i\Omega t}),$$
$$Y = \frac{i}{2}(-C + D - A\,e^{2i\Omega t} + B\,e^{-2i\Omega t}). \tag{8b}$$

Introducing the definitions

$$\phi_{\pm} \equiv \frac{1}{2}\bar{\omega}\sigma_{\pm}(\bar{\omega}), \qquad a_{\pm} \equiv \frac{\bar{\omega}}{4\Omega}[7\xi - \sigma_{\pm}(\bar{\omega})], \tag{8c}$$

the auxiliary quantities A, B, C, D are given by the following formulae:

$$A = (e^{i\phi_+ L} - e^{i(-2\Omega+\phi_-)L})\,e^{-i\theta}\frac{(1+a_-)a_+}{1+a_- - a_+},$$
$$B = (e^{i(2\Omega+\phi_+)L} - e^{i\phi_- L})\,e^{i\theta}\frac{(1-a_+)a_-}{1+a_- - a_+},$$
$$C = e^{i\theta+i(2\Omega+\phi_+)L}\frac{a_+ a_-}{1+a_- - a_+} + e^{i\theta+i\phi_- L}\frac{(1-a_+)(1+a_-)}{1+a_- - a_+},$$
$$D = e^{-i\theta+i(-2\Omega+\phi_-)L}\frac{a_+ a_-}{1+a_- - a_+} + e^{-i\theta+i\phi_+ L}\frac{(1-a_+)(1+a_-)}{1+a_- - a_+}. \tag{9a}$$

These formulae are accurate to first order in small quantities in the phases, and to zeroth order in small quantities in the real amplitudes multiplying the phases. As we stressed in section 1, first-order corrections to the real amplitudes are independent of L, and so can make no contribution to physical attributes, such as the ellipticity, of the emerging wave. Expanding the exponentials through first order in small quantities, we get the following expressions for A, B, C, D (with $\sigma_{\pm} \equiv \sigma_{\pm}(\bar{\omega})$):

$$A = i\bar{\omega}L\left[\frac{2\Omega}{\bar{\omega}} + \frac{1}{2}(\sigma_+ - \sigma_-)\right]e^{-i\theta}\frac{(1+a_-)a_+}{1+a_- - a_+},$$
$$B = i\bar{\omega}L\left[\frac{2\Omega}{\bar{\omega}} + \frac{1}{2}(\sigma_+ - \sigma_-)\right]e^{i\theta}\frac{(1-a_+)a_-}{1+a_- - a_+},$$
$$C = \left[1 + i\bar{\omega}L\left(\frac{2\Omega}{\bar{\omega}} + \frac{1}{2}\sigma_+\right)\frac{a_+ a_-}{1+a_- - a_+} + i\bar{\omega}L\frac{1}{2}\sigma_-\frac{(1-a_+)(1+a_-)}{1+a_- - a_+}\right]e^{i\theta},$$
$$D = \left[1 + i\bar{\omega}L\frac{1}{2}\sigma_+\frac{(1-a_+)(1+a_-)}{1+a_- - a_+} + i\bar{\omega}L\left(-\frac{2\Omega}{\bar{\omega}} + \frac{1}{2}\sigma_-\right)\frac{a_+ a_-}{1+a_- - a_+}\right]e^{-i\theta}. \tag{9b}$$

These equations can be greatly simplified by using the following three algebraic identities obeyed by the quantities $a_{\pm}$:

$$a_+ - a_- + 2a_+ a_- = 0,$$
$$\frac{(1-a_+)(1+a_-)}{1+a_- - a_+} = 1 - \frac{a_+ a_-}{1+a_- - a_+},$$
$$1 + a_- - a_+ = 1 + \frac{\bar{\omega}}{4\Omega}(\sigma_+ - \sigma_-). \tag{10a}$$

We then get the remarkably simple formulae

$$X = \cos\theta\left(1 + \tfrac{11}{4}\,\mathrm{i}\bar{\omega}L\xi\right) + \tfrac{3}{4}\,\mathrm{i}\bar{\omega}L\xi\,\cos(2\Omega t - \theta),$$
$$Y = \sin\theta\left(1 + \tfrac{11}{4}\,\mathrm{i}\bar{\omega}L\xi\right) + \tfrac{3}{4}\,\mathrm{i}\bar{\omega}L\xi\,\sin(2\Omega t - \theta),$$
$$\tag{10b}$$

which together with equation ($8a$) are our final result for the wave transmission problem. They will be used in the following section to calculate the ellipticity and major axis orientation of the emerging wave.

The vacuum with magnetic field is a specific instance of a generic weakly birefringent medium which has orthogonal polarization eigenmodes with refractive indices $n_\parallel, n_\perp$. Hence, with the replacements $\tfrac{7}{2}\xi \to n_\parallel - 1$, $2\xi \to n_\perp - 1$, $\tfrac{11}{2}\xi \to n_\parallel + n_\perp - 2$, and $\tfrac{3}{2}\xi \to n_\parallel - n_\perp$, the preceding discussion generalizes to the case in which such a generic medium is slowly rotated.

4. Ellipticity and polarization axis of the emerging wave

We proceed next to calculate the polarization parameters of the emerging wave, following the exposition of Born and Wolf [8]. Let us write

$$X = \cos\theta + \mathrm{i}\Delta_X^I \simeq \cos\theta\,\mathrm{e}^{\mathrm{i}\Delta_X^I/\cos\theta} \equiv a_1\,\mathrm{e}^{\mathrm{i}\delta_1},$$
$$Y = \sin\theta\,\mathrm{i}\Delta_Y^I \simeq \sin\theta\,\mathrm{e}^{\mathrm{i}\Delta_Y^I/\sin\theta} \equiv a_2\,\mathrm{e}^{\mathrm{i}\delta_2},$$
$$\tag{11a}$$

with $\Delta_{X,Y}^I$ being first-order small quantities that can be read off from equation ($10b$). Then the auxiliary angle α of Born and Wolf is given by

$$\tan\alpha = \frac{a_2}{a_1} = \tan\theta, \tag{11b}$$

which implies that $\alpha = \theta$ up to second-order corrections, and so to first order there is no rotation of the polarization axis of the beam. The second auxiliary angle χ of Born and Wolf is given by $\sin 2\chi = \sin 2\alpha \sin\delta$; since $\delta = \delta_2 - \delta_1 = \Delta_Y^I\big/\sin\theta - \Delta_X^I\big/\cos\theta$ is first-order small, to leading order (and allowing the ellipticity to carry a sign) we have

$$\Psi = \text{Ellipticity} = \tan\chi \simeq \chi \simeq \tfrac{1}{2}\delta\sin 2\alpha \simeq \tfrac{1}{2}\delta\sin 2\theta$$
$$= \cos\theta\,\Delta_Y^I - \sin\theta\,\Delta_X^I. \tag{11c}$$

From the values of $\Delta_{X,Y}^I$ obtained from equation ($10b$), we then find

$$\Psi = -\tfrac{3}{4}\xi\bar{\omega}L\sin 2(\theta - \Omega t), \tag{11d}$$

which using $\bar{\omega} = 2\pi/\lambda$ and $\xi = 2\Delta n/3$ is equivalent to

$$\Psi = -\frac{\pi L}{\lambda}\Delta n\sin 2(\theta - \Omega t). \tag{11e}$$

Thus, to first order in small quantities, the ellipticity is exactly what would be calculated by assuming the magnetic field to be frozen at its instantaneous position during its traversal by the laser beam, using the formula for ellipticity calculated from the vacuum birefringence in a static magnetic field.

5. Perturbation method for a rotating, time-dependent magnetic field

The simple form of the final answer of equation ($10b$) suggests there should be a simpler derivation than the wave matching procedure employed in section 3. We show in this section that the same answer can be obtained by a perturbation theory approach, which while not

yielding the structure of the propagation modes in the magnetic field region, has the advantage that it extends to the case when the magnetic field strength (that is, the coefficient ξ) is also time dependent. Substituting the general ansatz

$$\vec{e} = E_1(z,t)\hat{B} + E_2(z,t)\hat{z} \times \hat{B},$$
$$\vec{b} = B_1(z,t)\hat{B} + B_2(z,t)\hat{z} \times \hat{B}, \tag{12a}$$

into equation (1a), and using equation (1b) but now allowing ξ to have a t and z dependence, we get the coupled equations

$$\partial_z E_1(z,t) = -\Omega B_1(z,t) - \partial_t B_2(z,t),$$
$$-\partial_z E_2(z,t) = -\partial_t B_1(z,t) + \Omega B_2(z,t),$$
$$(1-6\xi)\partial_z B_1(z,t) = (1+5\xi)\Omega E_1(z,t) + (1-2\xi)\partial_t E_2(z,t) - 2\partial_t \xi E_2(z,t) + 6\partial_z \xi B_1(z,t),$$
$$-(1-2\xi)\partial_z B_2(t) = (1+5\xi)\partial_t E_1(z,t) - (1-2\xi)\Omega E_2(z,t) + 5\partial_t \xi E_1(z,t) - 2\partial_z \xi B_2(z,t). \tag{12b}$$

We can solve these equations by making a perturbation expansion in powers of ξ, by writing $E_{1,2} = E_{1,2}^{(0)} + E_{1,2}^{(1)} + \cdots$, and similarly for $B_{1,2}$, with the fields with subscript (0) the incident wave components on rotating bases,

$$E_1^{(0)} = \mathrm{e}^{\mathrm{i}\bar{\omega}(z-t)}\cos(\theta - \Omega t), \qquad E_2^{(0)} = \mathrm{e}^{\mathrm{i}\bar{\omega}(z-t)}\sin(\theta - \Omega t),$$
$$B_1^{(0)} = -\mathrm{e}^{\mathrm{i}\bar{\omega}(z-t)}\sin(\theta - \Omega t), \qquad B_2^{(0)} = \mathrm{e}^{\mathrm{i}\bar{\omega}(z-t)}\cos(\theta - \Omega t), \tag{13a}$$

and with the fields with superscript (1) perturbations proportional to ξ. The zeroth-order fields are exact solutions of equation (12b) when ξ is set to zero, but Ω is kept non-zero. Substituting the perturbation expansion into equation (12), differentiating $\partial_z E_{1,2}^{(1)}$ with respect to z to get $\partial_z^2 E_{1,2}^{(1)}$, and using equation (12b) to eliminate cross derivatives $\partial_t \partial_z B_{1,2}^{(1)}$ in terms of $E_{1,2}^{(1)}$, we get inhomogeneous wave equations for the first-order perturbations:

$$(\partial_z^2 - \partial_t^2)E_{1,2}^{(1)} = I_{1,2},$$
$$I_1 = 7\xi \partial_t^2 E_1^{(0)} + 12\partial_t \xi \partial_t E_1^{(0)} + 5\partial_t^2 \xi E_1^{(0)} - 2\partial_z \xi \partial_t E_1^{(0)} - 2\partial_z \partial_t \xi E_1^{(0)}, \tag{13b}$$
$$I_2 = 4\xi \partial_t^2 E_2^{(0)} + 2\partial_t \xi \partial_t E_2^{(0)} - 2\partial_t^2 \xi E_2^{(0)} - 6\partial_z \xi \partial_t E_2^{(0)} - 6\partial_z \partial_t \xi E_2^{(0)},$$

where we have dropped terms of order $\xi\Omega$ (but have kept Ω in the phases of the zeroth-order solution, where it multiplies t, which can be arbitrarily large). Using the Green's function $G(z,t)$ for the one-dimensional wave equation with outgoing wave boundary conditions

$$G(z,t) = -\tfrac{1}{2}\theta(t-|z|), \qquad (\partial_z^2 - \partial_t^2)G(z,t) = \delta(z)\delta(t), \tag{13c}$$

we can solve equation (13b) to give

$$E_{1,2}^{(1)}(z,t) = \int_0^L \mathrm{d}z' \int_{-\infty}^{\infty} \mathrm{d}t' G(z-z', t-t') I_{1,2}(z',t'). \tag{14}$$

A similar solution can be obtained for the perturbations $B_{1,2}^{(1)}$, giving the general first-order corrections to the fields, even when the magnetic field strength is t and z dependent.

Let us now consider the case when the field strength is only weakly varying in time, so that $\partial_t \xi/\xi$ is a small parameter. Then to first order in small quantities, the terms in equation (13b) involving time derivatives of ξ as well as explicit powers of Ω can be dropped, so that $I_{1,2}$ reduce to

$$I_1 = \mathrm{e}^{\mathrm{i}\bar{\omega}(z-t)}\cos(\theta - \Omega t)[-7\xi\bar{\omega}^2 + 2\mathrm{i}\bar{\omega}\partial_z \xi],$$
$$I_2 = \mathrm{e}^{\mathrm{i}\bar{\omega}(z-t)}\sin(\theta - \Omega t)[-4\xi\bar{\omega}^2 + 6\mathrm{i}\bar{\omega}\partial_z \xi]. \tag{15a}$$

When the magnetic field is uniform (although possibly weakly time dependent) in the interval $0 \leqslant z \leqslant L$, and zero elsewhere, one has $\partial_z \xi = \bar{\xi}(t)[\delta(z) - \delta(z - L)]$, with $\partial_t \bar{\xi}(t)$ second-order small. Then substituting equation (15a) into equation (14), taking $z = L$, so that $|z - z'| = L - z'$ on the support of the integrand, and regarding terms of order $\Omega/\bar{\omega}$ and ΩL as small, we can evaluate the integrals over z', t' to get the first-order fields. For the contribution from the terms $-(7, 4)\xi\bar{\omega}^2$ in the square brackets in equation (15a), which have support on the interval $0 \leqslant z' \leqslant L$, we use

$$\int_0^L dz' \int_{-\infty}^{\infty} dt' \theta(t - t' - L + z')\, e^{i\bar{\omega}(z'-t')\pm i\Omega t'} = \int_0^L dz' \int_{-\infty}^{t-L+z'} dt'\, e^{i\bar{\omega}(z'-t')\pm i\Omega t'}$$

$$\simeq \int_0^L dz'\, \frac{i}{\bar{\omega}}\, e^{i\bar{\omega}(L-t)\pm i\Omega(t-L+z')}$$

$$\simeq \frac{iL}{\bar{\omega}}\, e^{i\bar{\omega}(L-t)\pm i\Omega t}, \tag{15b}$$

while for the contribution from the terms $(2, 6)\,i\bar{\omega}\partial_z\xi$ in the square brackets in equation (15a), which have support on the field boundaries at $z' = 0, L$, we use

$$\int dz' \int dt'\,\theta(t - t' - L + z')\bar{\xi}(t')[\delta(z') - \delta(z' - L)]\, e^{i\bar{\omega}(z'-t')\pm i\Omega t'}$$

$$= \int_{-\infty}^{t} du\, e^{i\bar{\omega}(L-u)\pm i\Omega u}[\bar{\xi}(u - L)\, e^{\mp i\Omega L} - \bar{\xi}(u)]$$

$$= O(\partial_u\bar{\xi}, \bar{\xi}\Omega L) \simeq 0. \tag{15c}$$

Adding back the zeroth-order fields, we get for the fields at $z = L$,

$$E_1|_{z=L} = e^{i\bar{\omega}(z-t)} \cos(\theta - \Omega t)\left(1 + \tfrac{7}{2}i\bar{\omega}\xi L\right),$$

$$E_2|_{z=L} = e^{i\bar{\omega}(z-t)} \sin(\theta - \Omega t)\left(1 + \tfrac{4}{2}\,i\bar{\omega}\xi L\right), \tag{16a}$$

which when transformed back to the fixed bases by

$$E_1|_{z=L}(\cos\Omega t, \sin\Omega t, 0) + E_2|_{z=L}(-\sin\Omega t, \cos\Omega t, 0) = e^{i\bar{\omega}(z-t)}(X, Y), \tag{16b}$$

give

$$X = \cos\theta\left(1 + \tfrac{11}{4}\,i\bar{\omega}L\xi\right) + \tfrac{3}{4}\,i\bar{\omega}L\xi \cos(2\Omega t - \theta),$$

$$Y = \sin\theta\left(1 + \tfrac{11}{4}\,i\bar{\omega}L\xi\right) + \tfrac{3}{4}\,i\bar{\omega}L\xi \sin(2\Omega t - \theta), \tag{16c}$$

in agreement with our earlier result of equation (10b).

6. Discussion

Returning to equation (5b), we see that the form of the propagation eigenmodes in the magnetic field region is governed by the behaviour of the square root term in $\sigma_\pm$, that is, by $((2\Omega/\omega)^2 + (3\xi/2)^2)^{\frac{1}{2}}$, with distinctly different behaviour in the small Ω regime $2\Omega/\omega \ll 3\xi/2$, and the large Ω regime $2\Omega/\omega \gg 3\xi/2$. In the former, the refractive indices are $n_+ \simeq 1 + (7/2)\xi$ and $n_- \simeq 1 + 2\xi$, as in the case of a non-rotating magnetic field. In the latter, the refractive indices are $n_\pm \simeq 1 + (11/4)\xi \pm \Omega/\omega$. This change in the character of the eigenmodes does not show up in the transmitted wave polarization parameters, however, because all dependence on the square root cancels between the various terms in equation (11a). The simple results that we get for the polarization parameters are what one would immediately get by assuming that the dependence on ξ and Ω should be analytic around the origin in these parameters, since then the fact that effects of the magnetic field region vanish at $\xi = 0$ implies

that there can be no first-order term proportional to Ω; the rotation dependence must first enter at second order through terms proportional to $\xi\Omega$. Similarly, dependences on a small time rate of change of the magnitude of the magnetic field, which are relevant for experiments in which a non-rotating but ramping magnetic field is used, must also appear only in second-order terms in small parameters.

We note in conclusion that putting in the numbers characterizing the PVLAS experiment, even though their magnetic field rotates only a fraction of a revolution per second, their parameters yield $\Omega L = 0.7 \times 10^{-8}$, $2\Omega/\bar{\omega} \simeq 2 \times 10^{-15}$ and $3\xi/2 \simeq 10^{-22}$. So although all of these are small, PVLAS is in fact operating in the large Ω regime in terms of the behaviour of the square root and the propagation eigenmodes in the rotating field region. Nonetheless, as we have shown, the formulae they have used for the ellipticity and polarization axis orientation of the emerging wave are correct.

Acknowledgments

Note added. After this work was completed, we learned of related work by Biswas and Melnikov [9], whose results agree with ours.

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Corrigendum

Vacuum birefringence in a rotating magnetic field
Stephen L Adler 2007 *J. Phys. A: Math. Theor.* **40** F143–152

In the line above equation (1*d*), 'Maxwell equation' should read 'Maxwell equations'.
In equations (3*a*) and (3*b*), e^{iz} should read e^{ikz}. The wave number k appears correctly in subsequent equations.